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Strong Bogomolov inequality for stable vector bundles

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Abstract

Inspired by the recent conjectures concerning the existence of stable bundles on Calabi–Yau threefolds arising from string theory, we consider the possibility of strengthening the classical Bogomolov inequality. We show the existence of stable bundles violating such inequality on many complete intersections.
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1. Introduction

Let X be a smooth complex projective variety of dimension $n \geq 2$ and H an ample line bundle on X. It is a natural problem to determine for which $r \in \mathbb{Z}$ and $c_i \in H^{2i}(X, \mathbb{Z})$ there exists an H-stable vector bundle (or more generally torsion-free sheaf) E on X of rank r and the Chern classes $c_i(E) = c_i$. The problem has been investigated extensively in the case of algebraic surfaces [2,5,11]. By means of the elementary transformations, a general existence result in higher dimension has been obtained by Maruyama [6], under the assumption that $r \geq n$ and $c_2(E) \cdot H^{n-2}$ is sufficiently large. In [9,10] we considered the existence problem on Calabi–Yau manifolds, generalizing our earlier works on surfaces [7,8].

Recently, very interesting conjectures concerning sufficient conditions for the existence of stable bundles on Calabi–Yau threefolds have been proposed, which are inspired by superstring theory [3]. One of the conjectures states that for any smooth ample divisor D on a Calabi–Yau threefold X and $r \geq 2 \in \mathbb{Z}$, $c_i \in H^{2i}(D, \mathbb{Z})$, there exists a stable bundle E on D of rank r and Chern classes $c_i(E) = c_i$ if c_1 lifts to $H^2(X, \mathbb{Z})$ and the following inequality is satisfied:

$$2rc_2(E) - (r-1)c_1(E)^2 - \frac{r^2}{12}c_2(D) > 0.$$

Further, in the appendix of [3], the problem has been posed of whether on a simply connected surface D with ample or trivial canonical bundle, every stable bundle E with non-trivial moduli space obeys the inequality

$$2rc_2(E) - (r-1)c_1(E)^2 \geq \frac{r^2}{12}c_2(D)$$

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which may be considered as a strengthening of the well-known Bogomolov inequality:

$$2rc_2(E) - (r-1)c_1(E)^2 \geq 0.$$

The extra term $\frac{r^2}{12}c_2(D)$ here comes from an argument based on the attractor equations and its mathematical meaning is not clear to us at this moment. The purpose of this paper is to consider the possibility of such a strong Bogomolov inequality.

Let $\alpha(D, H)$ be a positive constant depending on a surface D and a polarization H on D. We say that the strong Bogomolov inequality of type α holds if every H-stable bundle E on D satisfies the inequality

$$2rc_2(E) - (r-1)c_1(E)^2 \geq r^2\alpha(D, H).$$

It turns out that, for a general complete intersection surface D whose Picard group is generated by an ample line bundle $\mathcal{O}_D(1)$, there exists a stable bundle E on D violating the inequality for any $\alpha > 2$. Thus we give a negative answer to the problem mentioned above. To prove this, we construct an infinite sequence $\{E_m\}_{m=1}^{\infty}$ of stable bundles of explicit ranks and Chern classes by means of the method exploited in [9,10], and compare the asymptotic behavior of the both sides of the inequality as m becomes large.

Similarly, for bundles on varieties X of dimension $n \geq 3$, we may define the strong Bogomolov inequality of the form

$$(2rc_2(E) - (r-1)c_1(E)^2) \cdot H^{n-2} \geq r^2\alpha(X, H).$$

It is known that one cannot choose

$$\alpha(X, H) = \frac{1}{12}c_2(X) \cdot H^{n-2}.$$

A counterexample has been given by M. Jardim for rank three bundles on a quintic hypersurface in $\mathbb{P}^4$ [3, Appendix B]. We notice that the stable bundles on Calabi–Yau threefolds constructed in [10] also give counterexamples of different types. Generalizing our examples, we shall show that the inequality of the above form fails in arbitrary dimension $n \geq 3$.

2. Construction of stable bundles

Let X be a smooth complex projective variety of dimension $n \geq 2$. Let H be an ample line bundle on X. The *minimal H-degree* $d_{\min}(H)$ is defined as follows.

$$d_{\min}(H) = \min\{L \cdot H^{n-1} \mid L \in \mathrm{Pic}(X), L \cdot H^{n-1} > 0\}.$$

A line bundle L on X is said to be *H-minimal* if $L \cdot H^{n-1} = d_{\min}(H)$. If X is a variety such that the Picard group $\mathrm{Pic}(X)$ is generated by an ample line bundle $\mathcal{O}_X(1)$, then $\mathcal{O}_X(1)$ itself is clearly $\mathcal{O}_X(1)$-minimal. The following result is essential for the construction of stable bundles in this paper.

Lemma 2.1. *Let X be a smooth projective variety such that $H^1(\mathcal{O}_X) = 0$. Let H be an ample line bundle on X. Let L be a line bundle on X such that there exists a divisor $D \in |L|$ which is smooth and irreducible. Let $\mathcal{L}$ be a line bundle on D and let $Q = \iota_*\mathcal{L}$, where $\iota : D \hookrightarrow X$ denotes the inclusion map. Let E be a coherent sheaf which fits in the non-split extension*

$$0 \to U \otimes \mathcal{O}_X \to E \to Q \to 0.$$

Then E is an H-stable torsion-free sheaf if the following conditions are satisfied.

(1) *L is H-minimal;*
(2) $\mathrm{Hom}(E, \mathcal{O}_X) = 0$.

Proof. First we notice that, since we have $\mathrm{Hom}(Q, \mathcal{O}_X) = 0$, the assumption (2) is equivalent to the condition that the natural map $U^\vee \to \mathrm{Ext}^1(Q, \mathcal{O}_X)$, obtained by applying $\mathrm{Hom}(\quad, \mathcal{O}_X)$ to the exact sequence defining E, is injective. Thus the lemma is a consequence of [9, Lemma 1.4] when Q is a torsion-free sheaf.

We prove the claim by induction on $r := \dim U$. Assume that $r = 1$ and let

$$0 \to \mathcal{O}_X \to E \to Q \to 0$$

be the non-trivial extension corresponding to an element $0 \neq \epsilon \in \mathrm{Ext}^1(Q, \mathcal{O}_X)$. Assume that E has a non-trivial torsion subsheaf $T \subset E$. Let f denote the composite map $T \hookrightarrow E \to Q$. We see that f must be injective, since if f had non-trivial kernel K, then K should be a torsion sheaf contained in $\mathcal{O}_X$, which is impossible. In particular, the image $\overline{T}$ of T by f is also non-trivial. We apply $\mathrm{Hom}(\quad, \mathcal{O}_X)$ to the exact sequence

$$0 \to \overline{T} \to Q \to Q/\overline{T} \to 0$$

and obtain the exact sequence

$$\cdots \to \mathrm{Ext}^1(Q/\overline{T}, \mathcal{O}_X) \to \mathrm{Ext}^1(Q, \mathcal{O}_X) \xrightarrow{\alpha} \mathrm{Ext}^1(\overline{T}, \mathcal{O}_X) \to \cdots.$$

Since f is injective, $\alpha(\epsilon) \in \mathrm{Ext}^1(\overline{T}, \mathcal{O}_X)$ vanishes. On the other hand, we notice that $\overline{T}$ must be of the form $\overline{T} = \iota_* \mathcal{F}$ for some rank one torsion-free sheaf $\mathcal{F}$ on D since D is smooth and irreducible. $\mathcal{F}$ can be written as $\mathcal{F} = \mathcal{I}_Z \otimes \mathcal{O}_D(-Y) \otimes \mathcal{L}$ where $\mathcal{I}_Z$ is the ideal sheaf of a closed subscheme Z of codimension at least two and Y is an effective divisor on D (Z, Y are possibly empty). It follows that $Q/\overline{T}$ has support of codimension at least two in X, so we have $\mathrm{Ext}^1(Q/\overline{T}, \mathcal{O}_X) = 0$. However, this implies that α is injective and hence $\epsilon = 0$, which is a contradiction. This shows that we have $T = 0$, that is, E is torsion-free. This proves the case $r = 1$.

Next we assume that the claim holds up to $r - 1$. Let E be a sheaf which fits in a non-split extension

$$0 \to U \otimes \mathcal{O}_X \to E \to Q \to 0$$

where U is a vector space with $\dim U = r$. Let $U_1 \subset U$ be a one dimensional subspace and let $\overline{U} := U/U_1$ and $\overline{E} := E/(U_1 \otimes \mathcal{O}_X)$. We have the following exact sequences

$$0 \to U_1 \otimes \mathcal{O}_X \to E \to \overline{E} \to 0 \tag{$*$}$$

and

$$0 \to \overline{U} \otimes \mathcal{O}_X \to \overline{E} \to Q \to 0. \tag{$**$}$$

Since the map $U^\vee \to \mathrm{Ext}^1(Q, \mathcal{O}_X)$ is injective by the assumption $\mathrm{Hom}(E, \mathcal{O}_X) = 0$, the maps $\overline{U}^\vee \to \mathrm{Ext}^1(Q, \mathcal{O}_X)$ and $U_1^\vee \to \mathrm{Ext}^1(\overline{E}, \mathcal{O}_X)$ are also injective. It follows from the inductive assumption and ($**$) that $\overline{E}$ is H-stable and torsion-free. Then we may apply [9, Lemma 1.4] to ($*$) and conclude that E is H-stable and torsion-free as desired. □

Let X be a smooth projective variety of dimension $n \geq 2$. Assume that there exists a divisor $D \in |L|$ which is smooth and irreducible and let $\iota : D \hookrightarrow X$ denote the inclusion. Let $\mathcal{L}$ be a line bundle on D which is generated by global sections. We extend the evaluation map $H^0(D, \mathcal{L}) \otimes \mathcal{O}_D \to \mathcal{L}$ to the map $\varphi : H^0(D, \mathcal{L}) \otimes \mathcal{O}_X \to \iota_* \mathcal{L}$. It is well-known that the kernel of φ is locally free and its dual $E(D, \mathcal{L})$ is called the *elementary transformation* of $H^0(D, \mathcal{L}) \otimes \mathcal{O}_X$ along $\mathcal{L}$.

Proposition 2.2. *Let X be a smooth projective variety of dimension $n \geq 2$ such that $H^1(\mathcal{O}_X) = 0$ and* $\mathrm{Pic}(X)$ *is generated by an ample line bundle $\mathcal{O}_X(1)$. Assume that there exists a divisor $D \in |\mathcal{O}_X(1)|$ which is smooth and irreducible and let $\mathcal{O}_D(m) := \mathcal{O}_X(m)_{|D}$. Then, for sufficiently large integer m, there exists an $\mathcal{O}_X(1)$-stable vector bundle E_m on X of rank $r_m := \chi(\mathcal{O}_D(m))$, the Euler characteristic of $\mathcal{O}_D(m)$, and Chern classes*

$$c_1(E_m) = \mathcal{O}_X(1), \qquad c_2(E_m) = m\mathcal{O}_X(1)^2.$$

Furthermore, E_m has non-trivial moduli space, that is, $\mathrm{Ext}^1(E_m, E_m) \neq 0$.

Proof. Since $\mathcal{O}_D(1)$ is ample, for sufficiently large m, $\mathcal{O}_D(m)$ is globally generated and $h^i(\mathcal{O}_D(m)) = 0$ for $i > 0$ by Serre's theorem. This yields $h^0(\mathcal{O}_D(m)) = \chi(\mathcal{O}_D(m))$. We set $E_m := E(D, \mathcal{O}_D(m))$ and $U_m := H^0(D, \mathcal{O}_D(m))$. Then E_m is a vector bundle of rank $r_m = h^0(\mathcal{O}_D(m))$ on X, which fits in the exact sequence

$$0 \to E_m^\vee \to U_m \otimes \mathcal{O}_X \to \iota_* \mathcal{O}_D(m) \to 0. \tag{$*$}$$

Applying $\mathcal{H}om_{\mathcal{O}_X}(\quad, \mathcal{O}_X)$ to (∗), we have the exact sequence

$$0 \to U_m^\vee \otimes \mathcal{O}_X \to E_m \to \iota_* \mathcal{O}_D(1-m) \to 0$$

by the isomorphisms $\mathcal{E}xt^1_{\mathcal{O}_X}(\iota_*\mathcal{O}_D(m), \mathcal{O}_X) \cong \iota_*\mathcal{O}_D(m)^\vee \otimes \mathcal{O}_X(1) \cong \iota_*\mathcal{O}_D(1-m)$. From this sequence and Lemma 2.1, we see that E_m is $\mathcal{O}_X(1)$-stable. From the sequence (∗) we have

$$\mathrm{ch}(E_m) = r_m \mathrm{ch}(\mathcal{O}_X) + \mathrm{ch}(\iota_*\mathcal{O}_D(1-m)).$$

Since the Grothendieck–Riemann–Roch formula yields

$$\mathrm{ch}(\iota_*\mathcal{O}_D(1-m)) = \iota_*(\mathrm{ch}(\mathcal{O}_D(1-m))\mathrm{td}(D))\mathrm{td}(X)^{-1},$$

we obtain

$$c_1(\iota_*\mathcal{O}_D(1-m)) = [D] = \mathcal{O}_X(1),$$
$$c_2(\iota_*\mathcal{O}_D(1-m)) = [D]^2 - \iota_*(c_1(\mathcal{O}_D(1-m))) = m\mathcal{O}_X(1)^2.$$

Hence E_m has the Chern classes

$$c_1(E_m) = \mathcal{O}_X(1), \qquad c_2(E_m) = m\mathcal{O}_X(1)^2.$$

We notice that

$$h^0(E_m) = h^0(\mathcal{O}_D(m)) = r_m$$

since $h^1(\mathcal{O}_X) = 0$ and $h^0(\mathcal{O}_D(1-m)) = 0$ for $m > 1$.

It remains to show $\mathrm{Ext}^1(E_m, E_m) \cong H^1(\mathcal{E}nd E_m) \neq 0$ for $m \gg 0$. We tensor the exact sequence (∗) with E_m and obtain the exact sequence

$$0 \to \mathcal{E}nd E_m \to U_m \otimes E_m \to \iota_*(E_m(m)_{|D}) \to 0.$$

This induces the exact sequence of cohomologies

$$0 \to \mathbb{C} \to U_m \otimes H^0(E_m) \to H^0(D, E_m(m)_{|D}) \to H^1(\mathcal{E}nd E_m) \to \cdots$$

since $h^0(\mathcal{E}nd E_m) = 1$. Assume that $\mathrm{Ext}^1(E_m, E_m) = 0$. Then we must have $h^0(E_m(m)_{|D}) = r_m^2 - 1$ since $h^0(E_m) = r_m$. We will show that this is impossible. The exact sequence

$$0 \to E_m(m-1) \to E_m(m) \to E_m(m)_{|D} \to 0$$

induces the sequence

$$0 \to H^0(E_m(m-1)) \to H^0(E_m(m)) \to H^0(E_m(m)_{|D}) \to \cdots.$$

This yields $h^0(E_m(m)) - h^0(E_m(m-1)) \leq h^0(E_m(m)_{|D}) = r_m^2 - 1$. We show that this is impossible. For $m \gg 0$, we have $h^1(\mathcal{O}_X(m)) = 0$. Thus from the exact sequence

$$0 \to U_m^\vee \otimes \mathcal{O}_X(m) \to E_m(m) \to \iota_*\mathcal{O}_D(1) \to 0$$

we obtain the exact sequence

$$0 \to U_m^\vee \otimes H^0(\mathcal{O}_X(m)) \to H^0(E_m(m)) \to H^0(D, \mathcal{O}_D(1)) \to 0.$$

Hence we obtain $h^0(E_m(m)) = r_m h^0(\mathcal{O}_X(m)) + h^0(\mathcal{O}_D(1))$. Similarly, the exact sequence

$$0 \to U_m^\vee \otimes \mathcal{O}_X(m-1) \to E_m(m-1) \to \iota_*\mathcal{O}_D \to 0$$

yields $h^0(E_m(m-1)) = r_m h^0(\mathcal{O}_X(m-1)) + 1$. Thus

$$h^0(E_m(m)) - h^0(E_m(m-1)) = r_m(h^0(\mathcal{O}_X(m)) - h^0(\mathcal{O}_X(m-1))) + h^0(\mathcal{O}_D(1)) - 1.$$

Since the sequence

$$0 \to \mathcal{O}_X(m-1) \to \mathcal{O}_X(m) \to \iota_*\mathcal{O}_D(m) \to 0$$

yields $h^0(\mathcal{O}_X(m)) - h^0(\mathcal{O}_X(m-1)) = h^0(\mathcal{O}_D(m)) = r_m$, we have

$$h^0(E_m(m)) - h^0(E_m(m-1)) = r_m^2 - 1 + h^0(\mathcal{O}_D(1)) > r_m^2 - 1,$$

which is a contradiction. Therefore we conclude that $\mathrm{Ext}^1(E_m, E_m) \neq 0$. □

3. Strong Bogomolov inequality

Let X be a smooth projective variety of dimension $n \geq 2$. We say that *the strong Bogomolov inequality of type* α holds, if there exists a constant $\alpha = \alpha(X, H) > 0$ depending on X and a polarization H on X, such that every H-stable bundle E on X, of rank r and Chern classes $c_i(E)$, satisfies

$$(2rc_2(E) - (r-1)c_1(E)^2) \cdot H^{n-2} \geq r^2\alpha.$$

In this section we shall give some examples of stable bundles violating such strong Bogomolov inequality. First we consider the case of surfaces.

Theorem 3.1. *Let X be a smooth projective surface such that $H^1(\mathcal{O}_X) = 0$ and $\mathrm{Pic}(X)$ is generated by an ample line bundle $\mathcal{O}_X(1)$. Assume that there exists a smooth divisor $C \in |\mathcal{O}_X(1)|$ and $\alpha = \alpha(X, \mathcal{O}_X(1)) > 2$. Then, for sufficiently large m, there exists an $\mathcal{O}_X(1)$-stable vector bundle E_m of rank r_m and Chern classes $c_i(E_m)$ on X, such that*

$$2r_mc_2(E_m) - (r_m - 1)c_1(E_m)^2 < r_m^2\alpha.$$

Proof. Let $K_X \cong \mathcal{O}_X(k)$ for an integer k and $d = \mathcal{O}_X(1)^2$. By the adjunction formula, C is a curve of genus $g := (k+1)d/2 + 1$ and the Riemann–Roch yields $\chi(\mathcal{O}_C(m)) = md + 1 - g$. By Proposition 2.2, there exists an $\mathcal{O}_X(1)$-stable bundle E_m of rank $r_m = md + 1 - g$ with Chern classes $c_i(E_m)$ satisfying

$$2r_mc_2(E_m) - (r_m - 1)c_1(E_m)^2 = 2d^2m^2 + (2 - 2g - d)dm + gd.$$

For any real number $\alpha > 2$, $r_m^2\alpha$ has the leading term $\alpha d^2m^2 > 2d^2m^2$. Hence, for $m \gg 0$, we have

$$2r_mc_2(E_m) - (r_m - 1)c_1(E_m)^2 < r_m^2\alpha. \quad \square$$

Corollary 3.2. *Let $X \subset \mathbb{P}^n$ be a general smooth complete intersection surface of type $(d_1, d_2, \ldots, d_{n-2})$. Assume that X is not one of the following types*

$$(2), (3), (4), (2, 2), (3, 2), (2, 2, 2)$$

and that

$$\left(\sum_{i=1}^{n-2} d_i - (n+1)\right) \prod_{i=1}^{n-2} d_i > 72.$$

Then, for sufficiently large m, there exists an $\mathcal{O}_X(1)$-stable vector bundle E_m on X, such that $\mathrm{Ext}^1(E_m, E_m) \neq 0$ *and*

$$2r_mc_2(E_m) - (r_m - 1)c_1(E_m)^2 < \frac{r_m^2}{12}c_2(X).$$

Proof. By the classical Noether–Lefschetz theorem [4], X is simply connected and $\mathrm{Pic}(X)$ is generated by $\mathcal{O}_X(1) = \mathcal{O}_{\mathbb{P}^N}(1)_{|X}$, if X is not one of the types listed in the corollary. Further, since the canonical bundle of X is given by

$$K_X = \mathcal{O}_X\left(\sum_{i=1}^{n-2} d_i - (n+1)\right),$$

K_X is ample and $c_2(X) \geq c_1(X)^2/3 = (\sum_{i=1}^{n-2} d_i - (n+1)) \prod_{i=1}^{n-2} d_i/3 > 24$ by the Miyaoka–Yau inequality. Hence the claim follows from Theorem 3.1. □

Let D be a general complete intersection surface of type $(5, d)$ in $\mathbb{P}^4$ for $d \gg 0$, which is an ample divisor with the ample canonical bundle of a quintic Calabi–Yau threefold. The assumptions of the corollary are satisfied for D, hence the strong Bogomolov inequality

$$2rc_2(E) - (r-1)c_1(E)^2 \geq \frac{r^2}{12}c_2(D)$$

does not hold. This answers negatively to a problem posed in [3, Appendix A].

Next we consider the case of dimension $n \geq 3$.

Theorem 3.3. *Let X be a smooth projective variety of dimension $n \geq 3$ such that $H^1(\mathcal{O}_X) = 0$ and* $\mathrm{Pic}(X)$ *is generated by an ample line bundle $\mathcal{O}_X(1)$ on X. Assume that there exists a divisor $D \in |\mathcal{O}_X(1)|$ which is smooth and irreducible. Then, for any $\alpha = \alpha(X, \mathcal{O}_X(1)) > 0$ and sufficiently large m, there exist $\mathcal{O}_X(1)$-stable bundles E_m on X of rank r_m and Chern classes satisfying*

$$(2r_m c_2(E_m) - (r_m - 1)c_1(E_m)^2) \cdot \mathcal{O}_X(1)^{n-2} < r_m^2\alpha.$$

Proof. Let $d := \mathcal{O}_X(1)^n$. As before, for $m \gg 0$ there exist stable bundles E_m of rank r_m, where

$$r_m = \chi(\mathcal{O}_D(m)) = \frac{d}{(n-1)!}m^{n-1} + O(m^{n-2})$$

and Chern classes $c_i(E_m)$ satisfying

$$(2r_m c_2(E_m) - (r_m - 1)c_1(E_m)^2) \cdot \mathcal{O}_X(1)^{n-2} = \frac{2d^2}{(n-1)!}m^n + O(m^{n-1}).$$

On the other hand,

$$r_m^2\alpha = \left(\frac{d}{(n-1)!}\right)^2 \alpha m^{2n-2} + O(m^{2n-3}).$$

Hence, for sufficiently large m, the claim holds. □

As a corollary, we obtain the following result.

Corollary 3.4. *Let $X \subset \mathbb{P}^N$ be a smooth complete intersection of dimension $n \geq 3$ with $c_2(X) \cdot \mathcal{O}_X(1)^{n-2} > 0$, where $\mathcal{O}_X(1) := \mathcal{O}_{\mathbb{P}^N}(1)_{|X}$. Then, for sufficiently large m, there exists an $\mathcal{O}_X(1)$-stable vector bundle E_m on X of rank r_m such that*

$$(2r_m c_2(E_m) - (r_m - 1)c_1(E_m)^2) \cdot \mathcal{O}_X(1)^{n-2} < \frac{r_m^2}{12}c_2(X) \cdot \mathcal{O}_X(1)^{n-2}.$$

We may compute r_m explicitly in case $n = 3$. Let $K_X = \mathcal{O}_X(k)$. Then, by the adjunction formula, $K_D = \mathcal{O}_D(k+1)$. By the Riemann–Roch formula on D, for $m \gg 0$, the rank r_m of E_m is given by

$$r_m = \chi(\mathcal{O}_D(m)) = \frac{m(m-k-1)d}{2} + \chi(\mathcal{O}_D).$$

By the exact sequence

$$0 \to \mathcal{O}_X(-1) \to \mathcal{O}_X \to \mathcal{O}_D \to 0,$$

we have $\chi(\mathcal{O}_D) = \chi(\mathcal{O}_X) - \chi(\mathcal{O}_X(-1))$. By the Riemann–Roch formula on X, we compute

$$\chi(\mathcal{O}_X(-1)) = \frac{-(k+1)(k+2)d}{12} - \frac{k+2}{24}c_2(X) \cdot \mathcal{O}_X(1).$$

Hence

$$r_m = \frac{m(m-k-1)d}{2} + \chi(\mathcal{O}_X) + \frac{(k+1)(k+2)d}{12} + \frac{k+2}{24}c_2(X) \cdot \mathcal{O}_X(1).$$

For example, let X be a quintic threefold in $\mathbb{P}^4$. Then bundles E_m are of rank

$$r_m = \frac{5m(m-1)}{2} + 5$$

and violate the strong Bogomolov inequality since $c_2(X) \cdot \mathcal{O}_X(1) = 50$. These bundles have been constructed in [10]. Another example has been given by Jardim [3, Appendix B].

We give examples of bundles on Calabi–Yau threefolds whose Picard group is not isomorphic to $\mathbb{Z}$, which have been considered in [10]. Let $X = \mathbb{P}^{(1,1,2,2,2)}$ [8] which is familiar in string theory [1]. Let $\widehat{X}$ denote the hypersurface of degree eight in the weighted projective space $\mathbb{P}^{(1,1,2,2,2)}$, which is defined by the equation

$$x_1^8 + x_2^8 + x_3^4 + x_4^4 + x_5^4 = 0.$$

Let $p : X \to \widehat{X}$ be the blow-up of $\widehat{X}$ along the curve of singularities and let E be the exceptional divisor. Then X admits a K3 fibration $\pi : X \to \mathbb{P}^1$. We let F denote a smooth fiber of π and let $H := 2F + E$. It is known that $\mathrm{Pic}(X)$ is generated by H and F. We fix an ample line bundle $H_q := H + qF$ for some $q \gg 0$. It is easy to see that F is H_q-minimal, hence by the argument as in the previous section, $E_m := E(F, mH_{|F})$ are H_q-stable bundles of rank $r_m = 2m^2 + 2$ and Chern classes

$$c_1(E_m) = F, \qquad c_2(E_m) = mH \cdot F$$

so that we have

$$(2r_m c_2(E_m) - (r_m - 1)c_1(E_m)^2) \cdot H_q = 16m^3 + 16m.$$

Since have $c_2(X) \cdot H_q = 24q + 56$,

$$\frac{r_m^2}{12} c_2(X) \cdot H_q = \frac{(4m^4 + 8m^2 + 4)(24q + 56)}{12}.$$

Thus these E_m violate the strong Bogomolov inequality.

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